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Rayleigh–Sommerfeld diffraction and Poisson’s spot

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Abstract

When the Fresnel–Kirchhoff (FK) diffraction integral is evaluated exactly (instead of using the Fresnel approximation), the well-known mathematical inconsistency in the FK boundary conditions leads to unacceptable results for the intensity of Poisson’s spot. The Rayleigh–Sommerfeld (RS) integral has no inconsistencies and leads to an accurate description. The case for RS is bolstered by the observation that it is equivalent to Fourier propagation.

1. Introduction

The boundary conditions imposed on the diffraction problem in order to obtain the Fresnel–Kirchhoff (FK) solution are well known to be mathematically inconsistent and to be violated by the solution when the observation point is close to the diffracting screen [1–3]. Most textbook treatments of diffraction acknowledge this fact but do not explore its implications. This leaves the student to wonder what those implications are and what limits they place on the applicability of the FK diffraction integral. In this paper, I show explicitly how the FK diffraction integral breaks down at points near the diffracting screen and that this problem is absent in the Rayleigh–Sommerfeld (RS) solution. This will aid the optics instructor who wants to explore diffraction theory in detail. I further aid the instructor by showing that Fourier propagation provides an alternative to the usual Green function approach to the derivation of the diffraction integral and leads directly to RS diffraction.

Describing Poisson’s spot is an excellent classroom means of illuminating the basic optical principles of diffraction and interference. I enhance the pedagogical value of Poisson’s spot by using it to discriminate between RS and FK diffraction and I provide a complete description of the on-axis and near-axis intensities of the Poisson spot diffraction pattern at all distances behind the obscuring disc.

The difference between RS and FK is in the inclination factor (also called the obliquity factor) and is normally immaterial because the inclination factor is normally approximated by 1. With this approximation, RS and FK are the same. But when the approximation is not valid, FK can lead to unacceptable answers. Calculating the on-axis intensity of Poisson’s spot

provides a critical test, a test passed by RS and failed by FK. FK fails because (a) convergence of the integral depends on how it is evaluated, and (b) when the convergence problem is fixed, the predicted amplitude at points near the obscuring disc is not consistent with the assumed boundary conditions.

Poisson's spot, also known as the spot of Arago, is the name given to the bright on-axis spot behind a circular obscuration illuminated by a plane wave: on the axis of the disc, all light diffracted at the rim of the disc arrives in phase and interferes constructively. Consequently, even for angles approaching 90° , i.e., for observation points close to the disc, diffraction can result in a significant intensity. (In the overwhelming majority of optics problems, diffraction at angles of more than a few degrees leads to vanishingly small intensities and can be safely ignored.) Treatments of Poisson's spot that make the Fresnel approximation [4, 5] do not apply to the region close to the disc. Treatments that do apply close to the disc use the RS integral [6] or its predecessor, the Rayleigh integral [7]. The RS integral is derived from the Rayleigh integral with the additional assumption that the wavelength of the light is small compared to the geometric dimensions of the problem, an assumption that is made throughout this paper. I refer to a calculation as exact to indicate that, while the short-wavelength approximation has been made, the Fresnel approximation has not. The other fundamental approximation made here is the scalar wave approximation, which means that the results apply better to acoustics than to optics, a point that will be reconsidered in the concluding remarks of section 5.

Fourier propagation provides an alternate means of handling diffraction problems. In this paper, the Fourier propagation theory is used to derive solutions in an integral form for the 2D (long slits and strips) and 3D (arbitrarily shaped apertures) diffraction problems. Fourier propagation reproduces the RS diffraction integral.

In section 2, I set up the basic diffraction problem and exhibit the difference between the RS and FK solutions. In section 3, I show the problem with FK in calculating the on-axis intensity of Poisson's spot, and then use RS to calculate the diffraction pattern at on- and off-axis points. In section 4, I show how Fourier propagation leads to the same solution of the diffraction problem as RS for the relatively simple 2D problem. Appendix B gives the more complex derivation for the 3D problem.

2. The basic diffraction problem

Solving the basic diffraction problem requires finding a solution to the Helmholtz equation for a propagating wave encountering a partially obscuring planar screen. The Helmholtz equation is

$$(\nabla^2 + k^2)U(x, y, z) = 0, \quad (1)$$

where $k = 2\pi/\lambda$ and U describes the amplitude and phase of the wave. U is a scalar, so only scalar diffraction theory is addressed. The boundary condition imposed on the solution to this differential equation is the effect of a diffracting screen in the $z = 0$ plane. Denoting by T the parts of the screen that are transmissive and by B the parts that block the beam, the boundary conditions used for the RS and FK solutions are

$$\begin{aligned} \text{RS and FK: } & U(x, y, 0) = U_0(x, y, 0) && \text{for } (x, y) \in T, \\ & U(x, y, 0) = 0 && \text{for } (x, y) \in B, \\ \text{FK only: } & \left. \frac{\partial U(x, y, z)}{\partial z} \right|_{z=0} = \left. \frac{\partial U_0(x, y, z)}{\partial z} \right|_{z=0} && \text{for } (x, y) \in T, \\ & \left. \frac{\partial U(x, y, z)}{\partial z} \right|_{z=0} = 0 && \text{for } (x, y) \in B, \end{aligned} \quad (2)$$

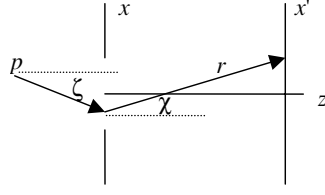


Figure 1. The basic diffraction problem: a point source, p , illuminates an aperture and produces a diffraction pattern on an observing screen. r is the distance from a point in the aperture to a point on the observing screen. ζ and χ are the angles the incident and the diffracting rays make with the z -axis.

where U_0 describes the incident wave and $\partial U/\partial z$ is the derivative of U normal to the diffracting plane.

With reference to figure 1, the RS diffraction integral [1, 2] for U at a distance z behind an aperture in a planar mask is

$$U_{\text{RS}}(x', y', z) = -\frac{i}{\lambda} \int_{\text{Area}} U_0(x, y, 0) \frac{\exp(ikr)}{r} \cos \chi \, dx \, dy, \quad (3)$$

where $r = [(x' - x)^2 + (y' - y)^2 + z^2]^{1/2}$ is the distance between $(x, y, 0)$ and (x', y', z) , χ is the diffraction angle at point $(x, y, 0)$, i.e., the angle the diffracted ray makes with the normal to the plane (not with the direction of the incoming wave), and the integral is over the area of the aperture. The FK integral [1, 3] is

$$U_{\text{FK}}(x', y', z) = -\frac{i}{\lambda} \int_{\text{Area}} U_0(x, y, 0) \frac{\exp(ikr)}{r} \frac{1}{2} (\cos \zeta + \cos \chi) \, dx \, dy, \quad (4)$$

where ζ is the incidence angle at point $(x, y, 0)$. The $\exp(ikr)/r$ factors in equations (3) and (4) express Huygens' principle: each point in the aperture acts as a source of spherical waves that combine to give the diffraction pattern. The cosine factors are called the inclination factors, and constitute the only difference between RS and FK. For most diffraction problems the inclination factor is approximated by 1, causing the difference between RS and FK to disappear.

Goodman [1] gives a succinct derivation of both RS and FK and discusses the difference between them. The derivations use different Green functions, which require different boundary conditions to reduce the Green theorem integral to the familiar diffraction integrals given in equations (3) and (4). The basic problem with FK is that the Green theorem integral cannot be evaluated unless the values of both U and $\partial U/\partial z$ are assumed to be zero on the obscuring part of the diffracting screen, but we know from analytic function theory that if both U and $\partial U/\partial z$ are zero over any region, then $U \equiv 0$ everywhere. Thus, the FK solution cannot be fully mathematically consistent and must therefore be suspected of not always (at least) giving the right answer. In section 3, I use Poisson's spot to show explicitly that the FK solution can violate the boundary conditions that were assumed for its derivation.

3. Calculating the intensity of Poisson's spot

For the simple case, shown in figure 2, of a plane wave impinging normally on a circular disc of radius a , $\cos \zeta = 1$ and $\cos \chi = z/r$. I will calculate the on-axis amplitude and intensity

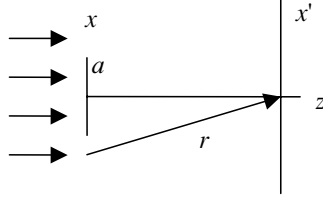


Figure 2. The diffraction problem for Poisson's spot: a plane wave falls on a circular disc of radius a . $r = [x^2 + y^2 + z^2]^{1/2}$ is the distance from a point in the plane of diffraction to a point on the axis of the disc.

behind the disc by doing so for an annular aperture and letting the outer radius of the annulus approach infinity:

$$U(0, 0, z) = -\frac{i}{\lambda} U_0 \int_{\rho=a}^{\rho \rightarrow \infty} \frac{\exp(ikr)}{r} \frac{1}{2} \left(c_1 + c_2 \frac{z}{r} \right) 2\pi \rho \, d\rho, \quad (5)$$

where $c_1 = 0$, $c_2 = 2$ for RS; $c_1 = c_2 = 1$ for FK and ρ is the radial coordinate in the (x, y) plane. In order to evaluate equation (5) without using the Fresnel approximation, I follow Sommerfeld [2] and Harvey *et al* [6] in changing the variable of integration from ρ to r . On the z axis $r = (z^2 + \rho^2)^{1/2}$, so $a \leq \rho \leq \infty \Rightarrow r_0 \leq r \leq \infty$, where $r_0 = (z^2 + a^2)^{1/2}$. Now $r^2 = z^2 + \rho^2$ so $r \, dr = \rho \, d\rho$, and the integral in equation (5) can be put in the right form to be evaluated via the Sommerfeld lemma given in appendix A:

$$\begin{aligned} U(0, 0, z) &= -ik U_0 \int_{r_0}^{r \rightarrow \infty} \exp(ikr) \frac{1}{2} \left(c_1 + c_2 \frac{z}{r} \right) dr \\ &= -ik U_0 \left[\frac{1}{ik} \exp(ikr) \frac{1}{2} \left(c_1 + c_2 \frac{z}{r} \right) \right]_{r=r_0}^{r \rightarrow \infty} + O\left(\frac{1}{k^2}\right) \\ &\approx U_0 \frac{1}{2} \left(c_1 + c_2 \frac{z}{r_0} \right) \exp(ikr_0) - U_0 \frac{c_1}{2} \exp[ik(r \rightarrow \infty)]. \end{aligned} \quad (6)$$

The RS version of equation (6) is

$$U_{\text{RS}}(0, 0, z) = U_0 \frac{z}{r_0} \exp(ikr_0), \quad (7)$$

while the FK version is

$$U_{\text{FK}}(0, 0, z) = U_0 \frac{1}{2} \left(1 + \frac{z}{r_0} \right) \exp(ikr_0) - U_0 \frac{1}{2} \exp[ik(r \rightarrow \infty)], \quad (8)$$

which shows that the FK integral fails to converge in this case. The reader's attention is called to the fact that if a is set to zero (i.e. $r_0 = z$), there is no obscuration and the right-hand side of equation (8) should be just $U_0 \exp(ikz)$ —which, because of the second term, it is not! The FK integral, when evaluated in this straightforward way, does not give an acceptable answer.

Eliminating the second term in equation (8) can be done in various ways. An artificial way would be to make the outer edge of the annular aperture an ellipse or some other shape instead of a circle. Then the rays diffracted from this edge do not arrive on the axis in phase and do not interfere constructively. A more sensible way is to impose Babinet's principle as a separate requirement (separate, because, as we have just seen, FK does not satisfy it unless the integral is done the right way). Babinet's principle requires that the sum of the obscuration diffraction pattern (figure 2) and the aperture diffraction pattern (figure 2 with the transmitting and blocking regions reversed) be the uninterrupted plane wave: $U_{\text{ob}} + U_{\text{ap}} = U_0 \exp(ikz)$.

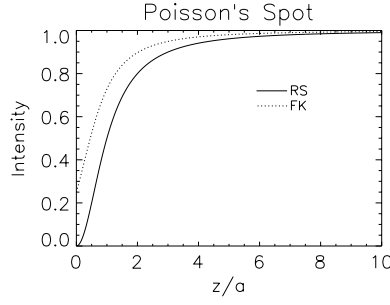


Figure 3. Intensity of Poisson's spot at on-axis distance z behind a disc of radius a , according to the Rayleigh–Sommerfeld and Fresnel–Kirchhoff diffraction theories.

Thus $U_{\text{ob}} = U_0 \exp(ikz) - U_{\text{ap}}$, which can be seen from equations (5) and (6) to be equation (8) without the second term:

$$\begin{aligned} U_{\text{FK}}(0, 0, z) &= U_0 \exp(ikz) + \frac{i}{\lambda} U_0 \int_0^a \frac{\exp(ikr)}{r} \frac{1}{2} \left(1 + \frac{z}{r}\right) 2\pi \rho \, d\rho \\ &= U_0 \frac{1}{2} \left(1 + \frac{z}{r_0}\right) \exp(ikr_0). \end{aligned} \quad (9)$$

The remaining, and more serious, problem with equation (9) is that it does not satisfy the boundary condition under which it was derived. As $z \rightarrow 0$, we should find $U(0, 0, z) \rightarrow 0$ in equations (7) and (9), since, from equation (2), that is the boundary condition originally assumed. Equation (7) satisfies this condition, while equation (9) does not. Also, omitting the $\exp(ika)$ phase factor, $\partial U / \partial z = U_0 / a$ at $z = 0$ for RS and half this for FK. The nonzero value of $\partial U / \partial z$ is consistent with RS, for which only U needs to be zero at the disc, but not for FK, which began with the additional boundary condition $\partial U / \partial z = 0$. Equations (7) and (9) are squared to obtain intensity and plotted in figure 3. The predicted intensities begin to differ appreciably at $z/a \approx 4$, where the diffraction angle is $\chi \approx 15^\circ$.

Having established the shortcomings of FK, I now use RS to calculate the off-axis intensity of Poisson's spot. The simplest way to calculate $U(x', y', z)$ for an off-axis point is to set $y' = 0$, calculate $U(x', 0, z)$, and then invoke symmetry to replace x' by ρ' , the polar coordinate in the (x', y') plane. For off-axis points with $y' = 0$, $r^2 = z^2 + (x - x')^2 + y^2 = z^2 + \rho^2 + x'^2 - 2x'\rho \cos \phi$, where $\rho^2 = x^2 + y^2$ and $x = \rho \cos \phi$ have been used. Therefore $r \, dr = (\rho - x' \cos \phi) \, d\rho \approx \rho \, d\rho$ as long as $\rho \geq a \gg x'$, which means that the following result is valid for points whose distance from the z -axis is small compared to the radius of the disc. Thus, I return to equation (3) and perform the integral over ρ again using $\rho \, d\rho = r \, dr$ and the Sommerfeld lemma:

$$\begin{aligned} U(x', 0, z) &= -\frac{i}{\lambda} U_0 \int_0^{2\pi} \int_a^\infty \frac{\exp(ikr)}{r} \frac{z}{r} \rho \, d\rho \, d\phi \approx -\frac{U_0}{2\pi} \int_0^{2\pi} \left[\exp(ikr) \frac{z}{r} \right]_{r=R(\phi)}^{r=\infty} d\phi \\ &= \frac{U_0}{2\pi} \int_0^{2\pi} \exp[ikR(\phi)] \frac{z}{R(\phi)} d\phi, \end{aligned} \quad (10)$$

where

$$\begin{aligned} R(\phi) &= \sqrt{z^2 + (x' - a \cos \phi)^2 + (a \sin \phi)^2} = \sqrt{z^2 + a^2 + x'^2 - 2x'a \cos \phi} \\ &\approx r_0 + \frac{x'^2}{2r_0} - \frac{x'a \cos \phi}{r_0} \end{aligned} \quad (11)$$

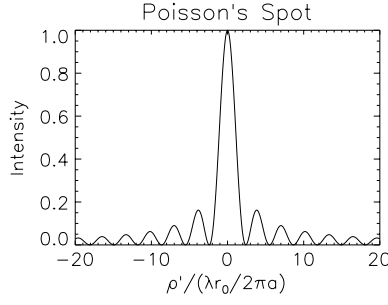


Figure 4. The off-axis intensity of Poisson's spot, relative to unit intensity on the axis (see the text).

Table 1. Some properties of J_0 and of Poisson's spot.

z	$J_0(z)$	$[2/(\pi z)]^{1/2} \cos(z - \pi/4)$	$J_0^2(z)$	Poisson's spot
0	1	Not applicable	1	Central intensity
2.40	0	-0.023	0	First dark ring
3.83	-0.402	-0.406	0.162	First bright ring
5.52	0	0.008	0	Second dark ring
7.02	0.300	0.301	0.090	Second bright ring
8.65	0	-0.003	0	Third dark ring
10.17	-0.248	-0.250	0.062	Third bright ring

is the distance between the points $(x, y, 0)$ and $(x', 0, z)$ and $x = a \cos \phi$, $y = a \sin \phi$ on the rim of the disc. I now substitute equation (11) into equation (10), keeping only the first term where $R(\phi)$ appears in the denominator:

$$\begin{aligned}
 U(x', 0, z) &\approx \frac{U_0}{2\pi} \int_0^{2\pi} \frac{z}{r_0} \exp \left[ik \left(r_0 + \frac{x'^2}{2r_0} - \frac{x'a \cos \phi}{r_0} \right) \right] d\phi \\
 &= U_0 \frac{z}{r_0} \exp \left[ik \left(r_0 + \frac{x'^2}{2r_0} \right) \right] \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left(-\frac{ikx'a \cos \phi}{r_0} \right) d\phi \\
 &= U_0 \frac{z}{r_0} \exp(ikr_0) \exp \left(\frac{i\pi x'^2}{\lambda r_0} \right) J_0 \left(\frac{2\pi x'a}{\lambda r_0} \right), \quad (12)
 \end{aligned}$$

where J_0 is the Bessel function, so

$$U(\rho', z) = U_{RS}(0, 0, z) \exp \left(\frac{i\pi \rho'^2}{\lambda r_0} \right) J_0 \left(\frac{2\pi \rho'a}{\lambda r_0} \right). \quad (13)$$

The relative intensity of light in the spot is given by the square of J_0 , which is plotted in figure 4.

Table 1 lists some of the properties of J_0 that are useful in this context. Some relevant mathematical relations are $dJ_0(z)/dz = -J_1(z)$, which will aid the reader in finding the extrema of J_0 ; $J_0(z) \approx [2/(\pi z)]^{1/2} \cos(z - \pi/4) = (\sin z + \cos z)/(\pi z)^{1/2}$, which is a good approximation for $z \geq 1$ (and only 15% too large at $z = 0.5$!), and $J_1(z) \approx [2/(\pi z)]^{1/2} \cos(z - 3\pi/4) = (\sin z - \cos z)/(\pi z)^{1/2}$. The bright peak at the centre of the diffraction pattern is Poisson's spot. Since $J_0(2.4) = 0$, we see that the radius of the first dark ring is $\rho'_1 = 2.4\lambda r_0/(2\pi a) = 0.38\lambda r_0/a$. The spacing of subsequent dark rings is closely approximated by $\Delta\rho' = 0.5\lambda r_0/a$, as shown in the table.

Note that the exactly on-axis intensity of Poisson's spot does not depend on wavelength because there is no wavelength dependence in the inclination factor. Wavelength dependence enters in the scaling of the radial intensity distribution: if the wavelength is doubled, the linear dimension of the diffraction pattern doubles, which means that it contains four times as much energy. Equation (13) applies for all values of z , but only for values of ρ' such that the approximation in equation (11) is valid. A more complete description requires the use of Lommel functions [4].

4. Doing diffraction with Fourier propagation

In section 4.1, I invoke the basic principles of Fourier propagation, which are used in section 4.2 to derive the RS diffraction integral. For ease of presentation, I address the 2D diffraction problem: long slits or strips illuminated by plane or cylindrical waves that can be described by $U(x, z)$. In section 4.1, the generalization to 3D is obvious; in section 4.2, it is done in appendix B.

4.1. Fourier propagation

The Fourier transform over x of $U(x, z)$ is called its angular spectrum, defined by

$$A(\alpha, z) \equiv \int_{-\infty}^{\infty} U(x, z) \exp(-ik\alpha x) dx. \quad (14)$$

Fourier propagation rests on the premise that, as shown by Goodman [1] for example,

$$A(\alpha, z) = A(\alpha, 0) \exp(ik\sqrt{1-\alpha^2}z), \quad (15)$$

where α is not restricted to $-1 \leq \alpha \leq 1$. The Fourier transform variable in equation (14) is α/λ , so the inverse transform of $A(\alpha, z)$ is

$$\begin{aligned} U(x', z) &= \int_{-\infty}^{\infty} A(\alpha, z) \exp(ik\alpha x') d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty} A(\alpha, 0) \exp(ik\sqrt{1-\alpha^2}z + ik\alpha x') d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, 0) \exp(-ik\alpha x) dx \exp(ik\sqrt{1-\alpha^2}z + ik\alpha x') d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, 0) \exp[ik\sqrt{1-\alpha^2}z + ik\alpha(x' - x)] dx d\left(\frac{\alpha}{\lambda}\right). \end{aligned} \quad (16)$$

The second equality in equation (16) shows that $U(x', z)$ is the sum of plane waves of amplitude $A(\alpha, 0)$, propagating at an angle $\theta = \cos^{-1} \alpha$ with respect to the x -axis. When $|\alpha| > 1$, the plane waves are evanescent, with exponentially decaying z -dependence given by $\exp[-(\alpha^2 - 1)^{1/2}kz]$. These waves do not propagate a significant distance from the aperture, but are needed to give a complete Fourier decomposition of $U(x', z)$ at, or near, $z = 0$.

We can easily check that $U(x', z)$ given by equation (16) is indeed a solution of the 2D Helmholtz equation. Since the dependence on x' and z is all in the exponent on the right-hand side of equation (16), we see that

$$\begin{aligned} \left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial z^2} \right) U(x', z) &= \left[(ik\alpha)^2 + (ik\sqrt{1-\alpha^2})^2 \right] U(x', z) \\ &= -k^2 U(x', z), \end{aligned} \quad (17)$$

as desired. Furthermore, equation (16) shows that the solution depends only on the value of U in the $z = 0$ plane. For the diffraction problem, the standard procedure is to use $U_0(x, 0)$ for $U(x, 0)$ in the transmissive parts of the screen and zero in the blocking parts. (Goodman loosely states that ‘Kirchhoff boundary conditions’ are applied [1], but actually only the RS conditions are needed.) Thus, Fourier propagation and the RS integral are solutions to the same differential equation with the same boundary condition, hence must be the same function, a fact that is shown explicitly in the next section.

4.2. The RS integral derived via Fourier propagation

Returning to equation (16), interchanging the order of integration, and using the RS boundary condition,

$$\begin{aligned} U(x', z) &= \int_{-\infty}^{\infty} U_0(x, 0) \int_{\alpha=-1}^{\alpha=1} \exp \{ik[\sqrt{1-\alpha^2}z + \alpha(x' - x)]\} d\left(\frac{\alpha}{\lambda}\right) dx \\ &\equiv \int_{-\infty}^{\infty} U_0(x, 0) \int_{\alpha=-1}^{\alpha=1} \exp[ikF(\alpha)] d\left(\frac{\alpha}{\lambda}\right) dx, \end{aligned} \quad (18)$$

where restricting the range of α to $-1 \leq \alpha \leq 1$ neglects the effects of evanescent waves, and the last line defines the function $F(\alpha)$. The α integral is done by the stationary phase method: the function $F(\alpha)$ is expanded in a Taylor series about the point α_0 at which its first derivative is zero. Using the notation $F_\alpha \equiv \partial F / \partial \alpha$, we require

$$F_\alpha(\alpha_0) = -\frac{\alpha_0 z}{\sqrt{1-\alpha_0^2}} + x' - x = 0, \quad (19)$$

which can be solved for α_0 ,

$$\alpha_0 = \frac{x' - x}{\sqrt{(x' - x)^2 + z^2}} = \frac{x' - x}{r_{2D}}. \quad (20)$$

Therefore

$$\sqrt{1-\alpha_0^2} = \frac{z}{r_{2D}} \quad (21)$$

and

$$F(\alpha_0) = \sqrt{1-\alpha_0^2}z + \alpha_0(x' - x) = r_{2D}. \quad (22)$$

The second derivative of $F(\alpha)$ at α_0 is

$$F_{\alpha\alpha}(\alpha_0) = -\frac{z}{\sqrt{1-\alpha_0^2}} - \frac{\alpha_0^2 z}{\sqrt{1-\alpha_0^2}^3} = -\frac{z}{\sqrt{1-\alpha_0^2}^3} = -\frac{r_{2D}^3}{z^2}. \quad (23)$$

Since the first-order term vanishes by construction, the Taylor series expansion through second order of $F(\alpha)$ about α_0 is

$$F(\alpha) \approx F(\alpha_0) + \frac{1}{2} F_{\alpha\alpha}(\alpha_0)(\alpha - \alpha_0)^2. \quad (24)$$

A standard Fresnel integral is now written in a form that will be useful below:

$$\int_{-\infty}^{\infty} \exp \left\{ -i \left[\frac{\pi}{\lambda} A(u - u_0)^2 \right] \right\} d\left(\frac{u}{\lambda}\right) = \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda A}}, \quad (25)$$

where A is positive definite.

The integral over α in equation (18) can now be evaluated. For ease of notation, drop the α_0 argument from $F_{\alpha\alpha}$ and observe that $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$ ($F_{\alpha\alpha}$ is always negative). Equation (24) is inserted into the exponent in equation (18) to obtain

$$\begin{aligned} \exp[ikF(\alpha_0)] \int_{\alpha} \exp \left\{ ik \left[-\frac{1}{2} |F_{\alpha\alpha}| (\alpha - \alpha_0)^2 \right] \right\} d \left(\frac{\alpha}{\lambda} \right) \\ = \exp(ikr_{2D}) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda|F_{\alpha\alpha}|}} = \exp(ikr_{2D}) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda r_{2D}}} \frac{z}{r_{2D}}. \end{aligned} \quad (26)$$

Therefore

$$\begin{aligned} U(x', z) &= \int_{-\infty}^{\infty} U_0(x, 0) \exp(ikr_{2D}) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda r_{2D}}} \frac{z}{r_{2D}} dx \\ &= \frac{1-i}{\sqrt{2\lambda}} \int_{-\infty}^{\infty} U_0(x, 0) \frac{\exp(ikr_{2D})}{\sqrt{r_{2D}}} \cos \chi_{2D} dx. \end{aligned} \quad (27)$$

Equation (27) is the 2D version of equation (3) and contains the 2D Huygens’ principle: each strip in the aperture acts as a source of cylindrical waves, for which amplitude falls off as $r_{2D}^{-\frac{1}{2}}$. The interested student may want to make the normal approximations [$\cos \chi_{2D} \approx 1$, $r_{2D} \approx z + (x' - x)^2/2z$ in the exponent, $r_{2D} \approx z$ outside it] and show that equation (27) reduces to the standard form that is evaluated with the Cornu spiral.

Extending equation (27) to the 3D problem will, first of all, add a dy to the integral. Keeping in mind that all the units of length must cancel out on the right-hand side, inspection of equation (27) suggests that the effect of a 3D calculation is to replace r_{2D} and χ_{2D} by r and χ , to replace cylindrical waves by spherical waves and to square the factor outside the integral. This intuitive argument is confirmed in detail in appendix B, with a result, equation (B.12), that matches the RS form of the diffraction integral given in equation (3).

5. Conclusion

The fundamental flaw in the Fresnel–Kirchhoff diffraction integral and the superiority of Rayleigh–Sommerfeld have been demonstrated with exact calculations of the intensity of Poisson’s spot. Fourier propagation has been presented as an alternate means of deriving the diffraction integral. Compared to the usual approach via Green’s theorem, this derivation has the advantage of rendering obvious the proper choice of boundary conditions. It has the disadvantage of requiring knowledge of Fourier propagation and, for the 3D version, more complicated maths, but the 2D version is not excessively difficult and the generalization to 3D by inspection is intuitively appealing.

As noted in the introduction, the argument has been confined to scalar wave theory, which will not be completely adequate for describing Poisson’s spot in optics for points near the disc. (The contribution of those rays not polarized parallel to the diffracting edge should be multiplied by the sine of the angle between the polarization vector and the direction to the observation point.) But it should be entirely adequate for describing Poisson’s spot in an acoustics experiment because acoustic waves are scalar (pressure) waves. Also, because the wavelength is much longer, diffraction phenomena can be more easily studied in an acoustics than in an optics laboratory. This experiment was done many years ago with somewhat equivocal results [8]. With modern equipment, it should not be particularly difficult to repeat, and could settle the conflict between RS and FK diffraction by direct measurement.

Appendix A. The Sommerfeld lemma

Following Sommerfeld [2] and Harvey *et al* [6], I perform a series of integrations by parts (only the first two are shown) to expand the integral of a function multiplied by a complex exponential in a series of terms:

$$\begin{aligned} \int_a^b f(x) \exp(ikx) dx &= \frac{f(x)}{ik} \exp(ikx) \Big|_a^b - \frac{f'(x)}{(ik)^2} \exp(ikx) \Big|_a^b + \frac{1}{(ik)^2} \int_a^b f''(x) \exp(ikx) dx \\ &\approx \frac{f(x)}{ik} \exp(ikx) \Big|_a^b, \end{aligned} \quad (\text{A.1})$$

where the approximation is justified if $f(x)$ is a slowly varying function, or, equivalently, in the limit $k \rightarrow \infty$ ($\lambda \rightarrow 0$). When applied to diffraction, the approximation in equation (A.1) holds when the geometric dimensions of a problem are large compared to a wavelength of light, otherwise $f(x)$ may not be sufficiently slowly varying.

The integral in equation (A.1) has exactly the form of the k -frequency coefficient of a Fourier series expansion of the function $f(x)$ over the interval (a, b) . This shows that for a large k we need to know only the values of $f(a)$ and $f(b)$ to find the value of the coefficient. I have searched more than a dozen Fourier series books and not found this lemma in any of them. It does appear (in a more general form) in Mandel and Wolf [9], who also give a thorough treatment of the stationary phase method of evaluating integrals.

Appendix B. Deriving the RS diffraction integral via Fourier propagation

Proceeding in analogy to section 4.2, the 3D version of equation (18) is

$$\begin{aligned} U(x', y', z) &= \iint_{x,y} U_0(x, y, 0) \iint_{\alpha,\beta} \exp \{ ik [\sqrt{1 - \alpha^2 - \beta^2} z \\ &\quad + \alpha(x' - x) + \beta(y' - y)] \} d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) dx dy \\ &\equiv \iint_{x,y} U_0(x, y, 0) \iint_{\alpha,\beta} \exp[ikF(\alpha, \beta)] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) dx dy. \end{aligned} \quad (\text{B.1})$$

The function $F(\alpha, \beta)$ is expanded in a Taylor series about the point (α_0, β_0) at which its first derivatives are zero. Setting

$$F_\alpha(\alpha_0, \beta_0) = -\frac{\alpha_0 z}{\sqrt{1 - \alpha_0^2 - \beta_0^2}} + x' - x = 0, \quad (\text{B.2})$$

and similarly for β , leads to

$$\alpha_0 = \frac{x' - x}{r}, \quad \beta_0 = \frac{y' - y}{r}, \quad \sqrt{1 - \alpha_0^2 - \beta_0^2} = \frac{z}{r}. \quad (\text{B.3})$$

The second derivatives of $F(\alpha, \beta)$ at (α_0, β_0) are

$$F_{\alpha\alpha}(\alpha_0, \beta_0) = -\frac{z}{\sqrt{1 - \alpha_0^2 - \beta_0^2}} - \frac{\alpha_0^2 z}{\sqrt{1 - \alpha_0^2 - \beta_0^2}^3} = -\frac{r[(x' - x)^2 + z^2]}{z^2}, \quad (\text{B.4})$$

$$F_{\beta\beta}(\alpha_0, \beta_0) = -\frac{r[(y' - y)^2 + z^2]}{z^2}, \quad (\text{B.5})$$

and

$$F_{\alpha\beta}(\alpha_0, \beta_0) = -\frac{\alpha_0\beta_0 z}{\sqrt{1 - \alpha_0^2 - \beta_0^2}^3} = -\frac{r(x' - x)(y' - y)}{z^2}. \quad (\text{B.6})$$

The following quantities will be needed below:

$$F(\alpha_0, \beta_0) = \frac{z^2}{r} + \alpha_0(x' - x) + \beta_0(y' - y) = r, \quad (\text{B.7})$$

and

$$F_{\alpha\alpha}(\alpha_0, \beta_0)F_{\beta\beta}(\alpha_0, \beta_0) - F_{\alpha\beta}^2(\alpha_0, \beta_0) = \frac{r^4}{z^2}. \quad (\text{B.8})$$

The Taylor series expansion through the second order of $F(\alpha, \beta)$ about the point (α_0, β_0) is

$$F(\alpha, \beta) \approx F(\alpha_0, \beta_0) + \frac{1}{2}F_{\alpha\alpha}(\alpha_0, \beta_0)(\alpha - \alpha_0)^2 + \frac{1}{2}F_{\beta\beta}(\alpha_0, \beta_0)(\beta - \beta_0)^2 + F_{\alpha\beta}(\alpha_0, \beta_0)(\alpha - \alpha_0)(\beta - \beta_0). \quad (\text{B.9})$$

The α and β integrals in equation (B.1) can now be evaluated. Observe that $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$ ($F_{\alpha\alpha}$ is always negative) and, for ease of notation, drop the (α_0, β_0) argument from the quantities $F_{\alpha\alpha}$, $F_{\beta\beta}$ and $F_{\alpha\beta}$. The first, second and fourth terms in equation (B.9) are inserted into the exponent in equation (B.1) and the integral over α evaluated by completing the square in the exponent [the symbol $\pm$ in equation (B.10) means add and subtract, not add or subtract]:

$$\begin{aligned} & \exp[ikF(\alpha_0, \beta_0)] \int_{\alpha} \exp \left\{ ik \left[-\frac{1}{2}|F_{\alpha\alpha}|(\alpha - \alpha_0)^2 + F_{\alpha\beta}(\alpha - \alpha_0)(\beta - \beta_0) \right] \right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr) \int_{\alpha} \exp \left\{ -\frac{ik}{2}|F_{\alpha\alpha}| \left[(\alpha - \alpha_0)^2 - 2\frac{F_{\alpha\beta}}{|F_{\alpha\alpha}|}(\alpha - \alpha_0)(\beta - \beta_0) \pm \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|^2}(\beta - \beta_0)^2 \right] \right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr) \exp \left[\frac{ik}{2} \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} (\beta - \beta_0)^2 \right] \\ & \quad \times \int_{\alpha} \exp \left\{ -\frac{ik}{2}|F_{\alpha\alpha}| \left[\alpha - \alpha_0 - \frac{F_{\alpha\beta}}{|F_{\alpha\alpha}|}(\beta - \beta_0) \right]^2 \right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr) \exp \left[\frac{ik}{2} \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} (\beta - \beta_0)^2 \right] \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda}|F_{\alpha\alpha}|}. \end{aligned} \quad (\text{B.10})$$

The integral over β in equation (B.1) can now be carried out by adding the third term in equation (B.9) to the exponent in equation (B.10) and again using $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$:

$$\begin{aligned} & \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda}|F_{\alpha\alpha}|} \exp(ikr) \int_{\beta} \exp \left(\frac{ik}{2} \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} (\beta - \beta_0)^2 + \frac{ik}{2} F_{\beta\beta} (\beta - \beta_0)^2 \right) d\left(\frac{\beta}{\lambda}\right) \\ &= \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda}|F_{\alpha\alpha}|} \exp(ikr) \int_{\beta} \exp \left[-\frac{ik}{2} \left(\frac{F_{\alpha\alpha}F_{\beta\beta} - F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} \right) (\beta - \beta_0)^2 \right] d\left(\frac{\beta}{\lambda}\right) \\ &= \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda}|F_{\alpha\alpha}|} \exp(ikr) \frac{1-i}{\sqrt{2}} \sqrt{\frac{|F_{\alpha\alpha}|}{\lambda(F_{\alpha\alpha}F_{\beta\beta} - F_{\alpha\beta}^2)}} = -\frac{iz}{\lambda r^2} \exp(ikr). \end{aligned} \quad (\text{B.11})$$

Equation (B.1) can now be written in the desired form:

$$\begin{aligned} U(x', y', z) &= -\frac{i}{\lambda} \iint_{x,y} U_0(x, y, 0) \frac{z \exp(ikr)}{r^2} dx dy \\ &= -\frac{i}{\lambda} \iint_{x,y} U_0(x, y, 0) \frac{\exp(ikr)}{r} \cos \chi dx dy. \end{aligned} \quad (\text{B.12})$$

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