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LETTERS AND COMMENTS

Comment on 'Rayleigh–Sommerfeld diffraction and Poisson's spot'

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Abstract
We show that some of the approximations made by Lucke (2006 *Eur. J. Phys.* **27** 193–204) in the analysis of diffraction by a circular disk are unnecessary.

In a recently published paper [1], the author solves for the diffraction of a normally incident plane wave by an opaque circular disc. Specifically, he obtains the field on axis in the shadow region using two different diffraction integrals, Fresnel–Kirchhoff (FK) and Rayleigh–Sommerfeld (RS), and he finds one of these approaches (RS) to be superior.

The RS analysis is based on several approximations.

(i) The scalar field U is used instead of the vector electric field $\vec{E}$.
(ii) An approximate form of the RS diffraction integral is applied.
(iii) The Kirchhoff approximation is invoked, i.e., the field appearing in the diffraction integral is assumed to be the field of the incident plane wave.
(iv) An approximation is made when evaluating the RS integral.

The purpose of this comment is to show that all but one of these approximations (iii) are unnecessary.

When the tangential components of the electric field are known on the plane $(x, y, 0)$, there is a rigorous, self-consistent form of Huygens' principle that obtains the electric field anywhere in the half space $z > 0$ [2]. For a point on the positive z axis, this formula reduces to

$$\vec{E}(0,0,z) = 2\mathrm{i}k \int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} \hat{r} \times [\hat{z} \times \vec{E}(\rho,\phi,0)] \left(1 + \frac{\mathrm{i}}{kr}\right)\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi. \quad (1)$$

The coordinates are shown in figure 1, and they are the same as those used in [1].

The radius of the opaque disc (perfectly conducting disc for the electromagnetic case) is a, and the electric field for the normally incident plane wave is

$$\vec{E}^{\mathrm{i}} = E_0\,\mathrm{e}^{\mathrm{i}kz}\hat{y}. \quad (2)$$

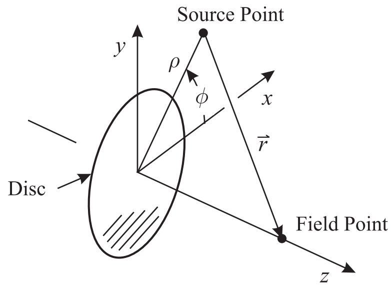

Figure 1. The coordinates used when analysing the diffraction from a disc.

Now invoking the Kirchhoff approximation (iii), the tangential component of the electric field on the plane $(x, y, 0)$ is taken to be

$$\begin{aligned}\vec{E}(\rho, \phi, 0) &= 0, \qquad \rho \leqslant a \\ &= E_0 \hat{y}, \qquad \rho > a.\end{aligned} \tag{3}$$

When (3) is inserted into (1) we obtain

$$\begin{aligned}\vec{E}(0,0,z) &= \frac{-\mathrm{i}kE_0}{2\pi}\int_{\rho=a}^{\infty}\int_{\phi=0}^{2\pi}(\rho\sin\phi\hat{z} + z\hat{y})\left(1+\frac{\mathrm{i}}{kr}\right)\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r^2}\right)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi \\ &= -\mathrm{i}kzE_0\int_{\rho=a}^{\infty}\left(1+\frac{\mathrm{i}}{kr}\right)\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{r^2}\right)\rho\,\mathrm{d}\rho\,\hat{y}.\end{aligned} \tag{4}$$

After changing the variable of integration to $r = \sqrt{\rho^2 + z^2}$ and letting $r_0 = \sqrt{a^2 + z^2}$, (4) becomes

$$\vec{E}(0,0,z) = -\mathrm{i}kzE_0\left(\int_{r=r_0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\mathrm{d}r + \int_{r=r_0}^{\infty}\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}kr}}{kr^2}\,\mathrm{d}r\right)\hat{y}. \tag{5}$$

Now the second integral can be performed using integration by parts to give our final answer:

$$\begin{aligned}\vec{E}(0,0,z) &= -\mathrm{i}kzE_0\left(\int_{r=r_0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\mathrm{d}r + \left[\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}kr}}{kr}\right]_{r_0}^{\infty} - \int_{r=r_0}^{\infty}\frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\mathrm{d}r\right)\hat{y} \\ &= E_0\left(\frac{z}{r_0}\right)\mathrm{e}^{\mathrm{i}kr_0}\hat{y}.\end{aligned} \tag{6}$$

Note that this result is consistent with our initial assumption that the tangential component of the electric field is zero on the disc, i.e., $\vec{E}(0,0,0) = 0$. It also gives the correct electric field in the limit $z \to \infty$, i.e., the incident field (2).

If we used the procedure presented above to analyse the complementary problem of a circular aperture in a perfectly conducting screen with the same illumination, we would obtain the following electric field:

$$\begin{aligned}\vec{E}_{\text{aperture}}(0,0,z) &= E_0\,\mathrm{e}^{\mathrm{i}kz}\hat{y} - E_0\left(\frac{z}{r_0}\right)\mathrm{e}^{\mathrm{i}kr_0}\hat{y} \\ &= \vec{E}^{\mathrm{i}} - \vec{E}_{\text{disc}}(0,0,z).\end{aligned} \tag{7}$$

As this result shows, the fields for the complementary problems, the disc and the aperture, are simply related. We might think that this is a consequence of Babinet's principle for electromagnetic fields, but it is not. Babinet's principle relates the electric field, $\vec{E}$, for one problem to the magnetic field, $c\vec{B}$, for the complementary problem [3]. The cause of the relationship in (7) is much simpler; it is a consequence of our use of the Kirchhoff approximation. The assumed electric fields on the plane $(x, y, 0)$ for the disc and aperture when added together produce exactly the incident electric field everywhere on this plane. Thus, the fields calculated from (1) in the half space $z > 0$ for these two problems when added together must equal the incident field.

In [1] additional approximations ((i), (ii) and (iv)) were used that were not introduced in the analysis presented here. Yet, the result obtained in [1] is the scalar version of (6):

$$U_{RS}(0, 0, z) = U_0 \left(\frac{z}{r_0}\right) \mathrm{e}^{\mathrm{i}kr_0}. \tag{8}$$

An obvious question is: why did these approximations not affect the result? The answer is that two of the approximations ((ii) and (iv)) made in [1] compensate for each other. To be more specific, the complete RS integral for this problem is [4, 5]

$$\begin{aligned} U_{\mathrm{RS}}(0, 0, z) &= -\mathrm{i}kzU_0 \int_{r=r_0}^{\infty} \left(1 + \frac{\mathrm{i}}{kr}\right) \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\, \mathrm{d}r \\ &= -\mathrm{i}kzU_0 \int_{r=r_0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\, \mathrm{d}r + kzU_0 \int_{r=r_0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr}}{kr^2}\, \mathrm{d}r, \end{aligned} \tag{9}$$

which in [1] is approximated (ii) by dropping the second integral:

$$U_{\mathrm{RS}}(0, 0, z) \approx -\mathrm{i}kzU_0 \int_{r=r_0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\, \mathrm{d}r. \tag{10}$$

Now integration by parts is used with (10) [1, appendix A] to obtain

$$U_{\mathrm{RS}}(0, 0, z) = U_0 \left(\frac{z}{r_0}\right) \mathrm{e}^{\mathrm{i}kr_0} - kzU_0 \int_{r=r_0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}kr}}{kr^2}\, \mathrm{d}r, \tag{11}$$

which in [1] is approximated (iv) by dropping the second integral:

$$U_{\mathrm{RS}}(0, 0, z) \approx U_0 \left(\frac{z}{r_0}\right) \mathrm{e}^{\mathrm{i}kr_0}. \tag{12}$$

Note that the terms dropped in approximating (9) and (11) would exactly cancel if kept, which means that these two approximations were unnecessary for obtaining the exact value of the integral (8).

In closing, we mention that the analysis presented here is for a time-harmonic field; however, it is easily adapted to the case of general time dependence, e.g., an incident field that is a pulse in time [6]. With a pulsed field, one can use the time of arrival to show that a portion of the field on axis arises from the edge of the disc (aperture).

References

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