OPTI 310, Spring 2020 **Problem Set 4** Prof. M. Kolesik

Due: Beginning of class, Wednesday Feb. 19 (15 points)

1. This problem deals with Maxwell and wave equations.

(a 1pt) Write down Maxwell's equations in the component form (i.e. showing explicitly all three components of both Ampere and Faraday, and expressing both divergence equations in terms of appropriate field components. Assume that the medium is a nonmagnetic dielectric with the refractive index n, $n^2 = \epsilon_r$. Allow for possible presence of free charges and currents.

(b 2pts) Starting from Maxwell's equations of a dielectric with relative permittivity ϵ_r, derive the wave equation for the magnetic field $\vec{B}(\vec{r}, t)$. This time assume that $\rho = 0$ and $\vec{J} = 0$. You may start from Maxwell equations written with the help of differential operator ∇, and follow a procedure similar to that we used in the class for the electric field. You should appreciate the symmetry between magnetic and electric fields.

(c 3pts) Demonstrate that the harmonic solution $\vec{B}(\vec{r}, t) = \hat{k}B_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t)]$ obeys your wave equation from part (b) provided $|\vec{k}| = n\omega/c$ and that (magnetic) Gauss requires $\vec{k}.\hat{k} = 0$. Recall that the former is an expression of the dispersion relation, while the latter says that the direction of the field oscillation is perpendicular to the wave propagation direction (transverse wave). Note: $\hat{k}$ is a coordinate unit vector while $\vec{k}$ is the wavevector.

(d 3pts) Consider propagation of the above wave in a medium with no free charges or currents. Using Ampere's law, calculate the electric vector field $\vec{E}(\vec{r}, t)$ that corresponds to $\vec{B}(\vec{r}, t)$ in part (c). Because we have not fully specified $\vec{k}$, you should obtain a formula valid for any $\vec{k}$ subject to conditions given in (c).

(e 1pt) Now assume that this wave propagates along negative x-axis. By sketching the orientation of the electric and magnetic vector directions, and the propagation vector for the above harmonic solutions demonstrate that their vector amplitudes form a right-handed set (specify their order).

2. This problem is related to the Poynting vector $\vec{S} = \frac{1}{\mu_0}(\vec{E}(\vec{r}, t) \times \vec{B}(\vec{r}, t))$, and to Gaussian beams.

(a 2pts) Consider the electric and magnetic fields for a harmonic wave propagating along the positive z-axis in a medium of refractive-index n

$$\vec{E}(\vec{r}, t) = \hat{\imath}E_0 \sin(kz - \omega t + \varepsilon), \quad \vec{B}(\vec{r}, t) = \hat{\jmath}B_0 \sin(kz - \omega t + \varepsilon), \tag{1}$$

where E_0 and B_0 are the peak amplitudes for the electric and magnetic fields, and Faraday's law tells us $E_0 = vB_0$, with $v = c/n$. By direct calculation show that the time-averaged Poynting vector for this harmonic wave solution is directed along the positive z-axis, and has an irradiance given by

$$I = \frac{1}{2}\epsilon_0 nc|\vec{E}_0|^2,$$

with $\vec{E}_0 = \hat{\imath}E_0$. (Hint: Neglect all fast-oscillating terms in the Poynting vector. Hint: Remember that Poynting is a vector, and you should always specify its direction, too.)

(b 1pt) A laser emits $P = 1$kW of power at a wavelength of $\lambda = 10\mu$m in a parallel Gaussian beam of cross-sectional area $A = 1$ cm^2. Assuming that the laser beam is Gaussian, calculate the peak amplitudes E_0 and B_0.

(c 1pt) Calculate the energy corresponding to a single photon, and the number of photons per second that the laser emits.

(d 1pt) Calculate the magnitude and describe the direction of the momentum of each photon emitted by the laser.