Problem Set # 1 – Solutions

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OPTI 210, Fall 2020 **Problem Set 1** Prof. M. Kolesik

Due: Beginning of class, Wednesday January 29th (20 points)

1. This problem deals with aspects of vector analysis. Three vectors are specified as follows: $\vec{A} = 2\hat{\imath} + 5\hat{\jmath}$, $\vec{B} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k}$, and $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath}$.

(a - 1pt) Given that the vectors $\vec{A}$ and $\vec{C}$ are orthogonal and that $\vec{C}$ is a unit vector, calculate values for C_x and C_y.

Recall

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$$

where θ is the angle between the vectors. Therefore, we use $cos(0) = 1$ for parallel vectors and $cos(90) = 0$ for orthogonal vectors.

Note: The familiar notation of

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1$$

is analogous to

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

Therefore:

$$\vec{A} \cdot \vec{C} = 2C_x + 5C_y$$

Given that $\vec{A}$ and $\vec{C}$ are orthogonal vectors, their dot product equals zero:

$$\vec{A} \cdot \vec{C} = 0 = 2C_x + 5C_y$$

Knowing this enables us to eliminate 1 unknown variable in terms of the other

$$C_x = (-5/2)C_y$$

Additionally, we are given that $\vec{C}$ is a unit vector so

$$C_x^2 + C_y^2 = 1$$

Plugging in

$${C_x}^2 = ((-5/2)C_y)^2 = (-5/2)^2{C_y}^2$$

To solve for C_y

$$(-5/2)^2{C_y}^2 + {C_y}^2 = 1$$
$$((25/4) + 4/4){C_y}^2 = 1$$
$${C_y}^2 = 4/29$$

Thus, $C_y = \pm\sqrt{4/29}$

In a similar manner, we can then plug this result into our original equation to eliminate C_y and solve for C_x algebraically Yielding $C_x = \mp\sqrt{25/29}, C_y = \pm\sqrt{4/29})$.

(b - 1pt) Without resort to calculation provide an example of a unit vector that is automatically orthogonal to both $\vec{A}$ and $\vec{C}$.

Both $\vec{A}$ and $\vec{C}$ are in the $\hat{\imath}$-$\hat{\jmath}$ plane so any vector that has only a component in the $\hat{k}$ direction will be orthogonal to $\vec{A}$ and $\vec{C}$. Since the vector we want has to have unit magnitude we just choose $\hat{k}$ or $-\hat{k}$.

Alternatively we could just construct a vector: $\vec{A} \times \vec{C}/|\vec{A} \times \vec{C}|$

(c - 1pt) Calculate the scalar product of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \cdot \vec{B} = (2)(1) + (5)(-4) + (0)(2) = 2 - 20 = -18$$

(d - 1pt) Calculate the cross product of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = [(5)(2) - (-4)(0)]\hat{\imath} + [(0)(1) - (2)(2)]\hat{\jmath} + [(2)(-4) - (5)(1)]\hat{k} = 10\hat{\imath} - 4\hat{\jmath} - 13\hat{k}$$

(e - 1pt) Calculate angle θ between $\vec{A}$ and $\vec{B}$.

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$$

so

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right)$$

Plugging in $|\vec{A}| = \sqrt{2^2 + 5^2} = \sqrt{29}, |\vec{B}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right) = \cos^{-1}\left(\frac{-18}{\sqrt{29}\sqrt{21}}\right) \approx 2.388 \text{ radians} \approx 136.8 \text{ degrees}$$

2. Scalar function h is given like so:

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

(a - 1pt) Calculate the gradient of h

The gradient of a scalar field is a (generally complex) vector field.

$$\nabla h = \hat{\imath}\partial_x h + \hat{\jmath}\partial_y h = 10(-6x + 2y - 18)\hat{\imath} + 10(-8y + 2x + 28)\hat{\jmath}$$

(b - 1pt) Calculate the Laplacian of h

To calculate the Laplacian, we perform a dot product between the del operator and our result from part (a). Since we are then calculating the divergence of a vector field, we obtain a scalar field result.

$$\nabla \cdot \nabla h = \partial_x^2 h + \partial_y^2 h = \partial_x(10(-6x + 2y - 18)) + \partial_y(10(-8y + 2x + 28)) = -60 + -80 = -140$$

3. Vector function $\vec{F}$ is given like so:

$$\vec{F}(x, y) = \left(\frac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2)^k\right)\hat{\imath} - \left(\frac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2)^k\right)\hat{\jmath},$$

(a - 1pt) Calculate divergence of $\vec{F}$

We can solve this using either by using a substitution before our operations or via direct computation. Although the former makes solving part (a) a bit more complex, if you use this method it will make part (b) simpler. Alternatively if you use the latter, straight forward method, part b will be more complicated.

Method 1: Let's start by rewriting $\vec{F}$ as:

$$\vec{F}(x, y) = yr^{2k-1}\hat{\imath} - xr^{2k-1}\hat{\jmath}$$

With the substitution $r = \sqrt{x^2 + y^2}$ because this makes sense geometrically (r is the radius in the x-y plane). Applying the chain rule:

$$\partial_x r^n = \frac{2xnr^{n-1}}{2r} = xnr^{n-2} \text{ and } \partial_y r^n = \frac{2ynr^{n-1}}{2r} = ynr^{n-2}$$

Thus, we can calculate $\nabla \cdot \vec{F}$ as:

$$\nabla \cdot \vec{F} = (\partial_x\hat{\imath} + \partial_y\hat{\jmath}) \cdot (yr^{2k-1}\hat{\imath} - xr^{2k-1}\hat{\jmath}) = (2k - 1)xyr^{2k-3} - (2k - 1)xyr^{2k-3} = 0$$

Method 2: Alternatively you can just use the straight forward method of applying the chain rule:

$$\nabla \cdot \vec{F} = y\partial_x(x^2 + y^2)^{k-1/2} - x\partial_y(x^2 + y^2)^{k-1/2}$$

which simplifies to

$$\nabla \cdot \vec{F} = (2k - 1)yx(x^2 + y^2)^{k-3/2} - (2k - 1)yx(x^2 + y^2)^{k-3/2} = 0$$

(b - 1pt) Calculate curl of $\vec{F}$ (Note: think before calculating, you may want to simplify first.)

$$\nabla \times \vec{F} = (\partial_y F_z - \partial_z F_y)\hat{\imath} + (\partial_z F_x - \partial_x F_z)\hat{\jmath} + (\partial_x F_y - \partial_y F_x)\hat{k} = (\partial_x F_y - \partial_y F_x)\hat{k}$$

So now we just need to do some product rule work:

Using Method 1:

$$\nabla \times \vec{F} = [\partial_x(-xr^{2k-1}) - \partial_y(yr^{2k-1})]\hat{k} = -r^{2k-1} - (2k-1)x^2 r^{2k-3} - r^{2k-1} - (2k-1)y^2 r^{2k-3}\hat{k}$$

Which simplifies to:

$$\nabla \times \vec{F} = -[2r^{2k-1} + (2k-1)(x^2+y^2)r^{2k-3}] = -(2k+1)r^{2k-1}\hat{k}$$

Using Method 2:

$$\nabla \times \vec{F} = -(2k+1)(x^2+y^2)^{k-1/2}\hat{k}$$

(c - 1pt) Decide if $\vec{F}$ has a potential, i.e. if there exists a function U such that $\vec{F} = \nabla U$.

Suppose that $\vec{F}$ had a potential U such that $\vec{F} = \nabla U$. For reference:

$$\vec{F}(x,y) = \left(\frac{y}{\sqrt{x^2+y^2}}(x^2+y^2)^k\right)\hat{\imath} - \left(\frac{x}{\sqrt{x^2+y^2}}(x^2+y^2)^k\right)\hat{\jmath}$$

Then

$$\nabla \times \vec{F} = \nabla \times \nabla U(x,y) = -(2k+1)(x^2+y^2)^{k-1/2}\hat{k}$$

For there to be a U $\vec{F}$ would need to be a conservative vector field. For this to happen, $k = -1/2$. For $k = -1/2$:

$$\vec{F}(x,y) = y/(x^2+y^2)\hat{\imath} - x/(x^2+y^2)\hat{\jmath}$$

So now if U exists, then $U = \int y/(x^2+y^2)dx = \tan^{-1}(x/y) + C$ because $\partial_x U = F_x$. However U must also be $\int -x/(x^2+y^2)dy = -\tan^{-1}(x/y) + C$ because $\partial_y U = F_y$. These two statements can not simultaneously be true, therefore there is no such U.

4. This problem deals with the use of complex numbers and the Euler formula

(a - 1pt) Calculate the magnitude and phase of the complex number $z = 9 - 12i$.

The magnitude of a complex number z is $|z| = \sqrt{z^* z}$. For arbitrary but real a and b, with $z = a + ib$,

$(a - ib)(a + ib) = a^2 + iab - iab + b^2 = a^2 + b^2 > 0$

Here we just need

$$z^* z = 9^2 + (-12)^2 = 81 + 144 = 225, \quad |z| = \sqrt{225} = 15$$

For the phase we just need to understand that z can be expressed in the form $z = Ae^{i\theta}$ for real $A > 0$. We know that $A = 15$ and that the real part is 9. This means:

$$15\cos(\theta) = 9, \theta = \cos^{-1}(9/15) \approx -0.927 \text{ rad} \approx -53.13 \text{ degrees}$$

We know the minus sign has to be there because the y component is negative (-12)

(b - 1pt) Show that the magnitude of a complex number in the form

$$z = \frac{a+ib}{a-ib}$$

is always equal to one for for real a and b. We shall encounter such form for the reflection coefficient in the total internal reflection...

$$|z|^2 = z^* z = \frac{a-ib}{a+ib}\frac{a+ib}{a-ib} = \frac{a^2+b^2}{a^2+b^2} = 1$$

(c - 1pt) By expressing the complex function $Ae^{-i\omega t}$ (ω is real) in terms of its real and imaginary parts, with $A = A_r + iA_i$ complex, show that the magnitude of the complex function is given by $|A|$.

$$Ae^{-i\omega t} = \underbrace{Acos(-\omega t)}_{A_r} + i\,\underbrace{Asin(-\omega t)}_{A_r}$$

$$|Ae^{-i\omega t}| = \sqrt{A^2[cos^2(-\omega t)^2 + \sin^2(-\omega t)]} = |A|$$

(d - 2pts) Using complex notation, prove that $\sin(3\theta) = [3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)]$, (Hint: You will want to make use of this: $e^{3i\theta} = (e^{i\theta})^3$).

We start by noticing that $\sin(3\theta)$ is the imaginary part of $e^{i3\theta}$. The hint tells us that $e^{3i\theta} = (e^{i\theta})^3$ and we are looking to get an expression purely in terms of sines and cosines of θ. Therefore we start by expanding:

$$e^{i3\theta} = \cos(3\theta) + i\sin(3\theta) = e^{i\theta^3}$$

$$e^{i\theta^3} = (\cos(\theta) + i\sin(\theta))^3 = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Now looking at only the imaginary part we have:

$$\sin(3\theta) = [3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)]$$