Problem Set # 2 – Solutions

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OPTI 210, Spring 2020 **Problem Set 2** Prof. M. Kolesik

Due: Beginning of class, Wed. Feb. 5th (15 points)

1. Consider a pulsed (i.e. finite-duration) wave of the form

$$\psi(x,t) = exp[-(x - 15 + vt)^2]cos(10(x + vt))$$

This function may approximate, e.g., an ultrafast optical pulse with a gaussian envelope.

(A - 4pts) Write a Matlab (or other language) code to plot this solution over the range $x = [-20, 25]$ for the parameters $v = 1$, and $t = 0, 15, 30$. Use different plotting colors for each time so that you can see how the solution evolves with time, and note the value of x at which the function is peaked for each time t. Please submit a listing of your code along with your numerical plot. Make sure that your plot has a decent resolution.

See final pages for matlab code and plots

(B - 1pts) Identify which of the two functions that constitute this pulsed waveform should be identified as the (faster changing) carrier wave and which plays the role of the (slower changing) envelope.

$$\boxed{\psi(x,t) = \underbrace{\exp[-(x - 15 + vt)^2]}_{\textit{Envelope}}\underbrace{\cos(10(x + vt))}_{\textit{Carrier}}}$$

2. This question deals with the harmonic wave solution

$$\psi(x,t) = A\sin(kx \pm \omega t + \pi/2),$$

where A is the amplitude of the wave form of propagation number k, and angular temporal frequency ω.

(A - 2pts) Show by direct evaluation that for the harmonic wave solution above

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi, \qquad \frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi.$$

$\partial_x^2 \sin(\alpha x) = \partial_x \alpha(\cos(\alpha x)) = -\alpha^2 \sin(\alpha x)$, therefore both of the above follow:

$$\frac{\partial^2}{\partial x^2} A \sin(kx \pm \omega t + \pi/2) = +kA\frac{\partial}{\partial x}\cos(kx \pm \omega t + \pi/2) = -k^2 sin(kx \pm \omega t + \pi/2)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$$

$$\frac{\partial^2}{\partial t^2} A \sin(kx \pm \omega t + \pi/2) = \pm\omega A\frac{\partial}{\partial t}\cos((kx \pm \omega t + \pi/2)) = -\omega^2 A \sin(kx \pm \omega t + \pi/2)$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$$

(B - 1pt) Using the results from part (A) show that demanding that the harmonic wave solution obeys the 1D wave equation requires that the *dispersion relation* $\omega = vk$ is satisfied, which relates the wave (phase) velocity, propagation number, and angular temporal frequency.

The harmonic wave equation states:

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

This is equivalent to

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

Plugging in $\psi(x,t)$:

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2\psi = \frac{-\omega^2}{v^2}\psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

So $-k^2\psi = \frac{-\omega^2}{v^2}\psi$ lets us cancel out the ψs and gets us $\omega = vk$

(C - 1pt) For which sign in front of ω does this wave propagate to the left?

Imagine sitting on top of a crest (a point with fixed or constant phase value) of a sine wave.

If it is moving to the left, the x value will be negative and decreasing. If $k > 0$, the kx term will therefore provide a negative contribution to the overall phase that decreases with the evolution of space (x) and time (t).

To maintain a constant phase, we need another term to provide a positive increase to the phase, with the evolution of space (x) and time (t,) that is equal in magnitude to the negative decrease to the phase from the kx term.

Therefore, the sign in front of the ωt must be positive since both time and frequency are positive.

(D - 1pt) For the wavelength range $\lambda = 300 - 700$ nm, and $v = 3 \times 10^8$ m/s, what is the corresponding ranges of spatial frequencies in m^{-1}, angular temporal frequencies in rads^{-1}, and temporal frequencies in s^{-1}.

Spatial Frequencies
$\kappa = 1/\lambda$ so $\kappa_{max} = 1/(300nm) = 3.33x10^6\text{m}^{-1}$, $\kappa_{min} = 1/(700nm) = 1.429x10^6\text{m}^{-1}$

Angular Temporal Frequencies
$\omega = 2\pi c/\lambda$ so $\omega_{min} = 2.6928 \times 10^{15}$ rad/s, $\omega_{max} = 6.2831 \times 10^{15}$ rad/s

Temporal Frequencies
$f = \omega/2\pi$ so $f_{min} = \omega_{min}/2\pi = 4.2857 \times 10^{14}$ Hz $= 429$ THz, $\omega_{max} = k_{max} * v = 1.00 \times 10^{15}$ Hz $= 1.00$ PHz

3. Here we consider the interference between two harmonic waves at $t = 0$

$$\psi(x, t = 0) = \sin(k_1 x) + \sin(k_2 x). \tag{1}$$

(A - 1pt) Using Matlab produce a plot of $\psi(x, t = 0)$ according to Eq. (1) on the interval $x = [0, 30]$ for $k_1 = 2\pi, k_2 = 1.8\pi$.

See final pages for matlab code and plots

(B - 2pts) By using the rules for the sum of trigonometric functions show that

$$\psi(x, t = 0) = 2\sin(K_+ x)\cos(K_- x), \tag{2}$$

with $K_\pm = (k_1 \pm k_2)/2 > 0$. The interference of the two waves can therefore be expressed as a product of cosine functions with different propagation numbers $K_\pm$. Hint: To apply one of the commonly listed Sum-to-Product trig identities, you may want to express $\sin(k_2 x)$ in term of cos function with a shifted argument.

Using

$$\sin(a) + \sin(b) = 2\sin((a + b)/2)\cos((a - b)/2)$$

We find

$$\sin(k_1 x) + \sin(k_2 x) = 2\sin((k_1 x + k_2 x)/2)\cos((k_1 x - k_2 x)/2$$

Yielding

$$\psi(x, t = 0) = 2\sin(K_+ x)\cos(K_- x)$$

(C - 2pts) For $k_1 \approx k_2$ the *average* propagation number $K_+ = (k_1 + k_2)/2 \approx k_1$ controls the fast carrier oscillations that arise in your plot from part (A), whereas the *"spread"* $K_- = (k_1 - k_2)/2 << K_+$ describes the slower spatial envelope that modulates the carrier oscillation. The spatial periods $L_\pm$ of the carrier and envelope fringes follow from the relations $K_\pm L_\pm = 2\pi$, which requires the harmonic waves to repeat in phase after a period. Demonstrate that the spatial periods $L_+ \approx 2\pi/k_1$ and $L_- \approx 4\pi/(k_1 - k_2)$ indeed appear in your numerical results from part (A).

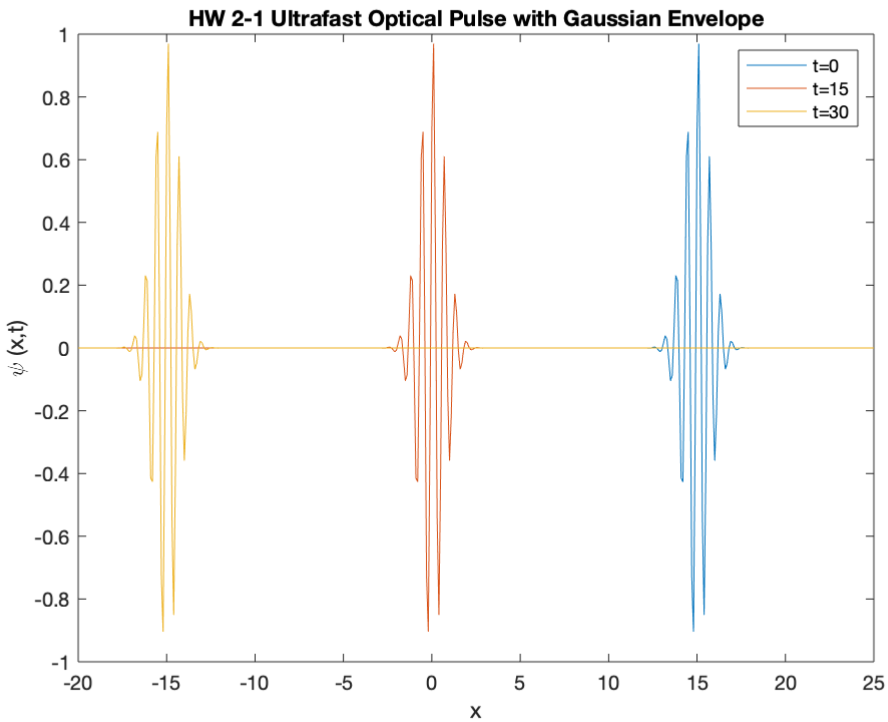
It is critical here to calculate $L_- = 20$ and $L_+ = 1.053 \approx 1$ and identify this on the plot.

$$\psi(x, t = 0) = \underbrace{2\cos(K_- x)}_{\text{Envelope}}\underbrace{\sin(K_+ x)}_{\text{Carrier}}$$

See plot on the following page of $\psi(x, 0)$ with carrier and envelope identified.

HW 2: Problem 1:

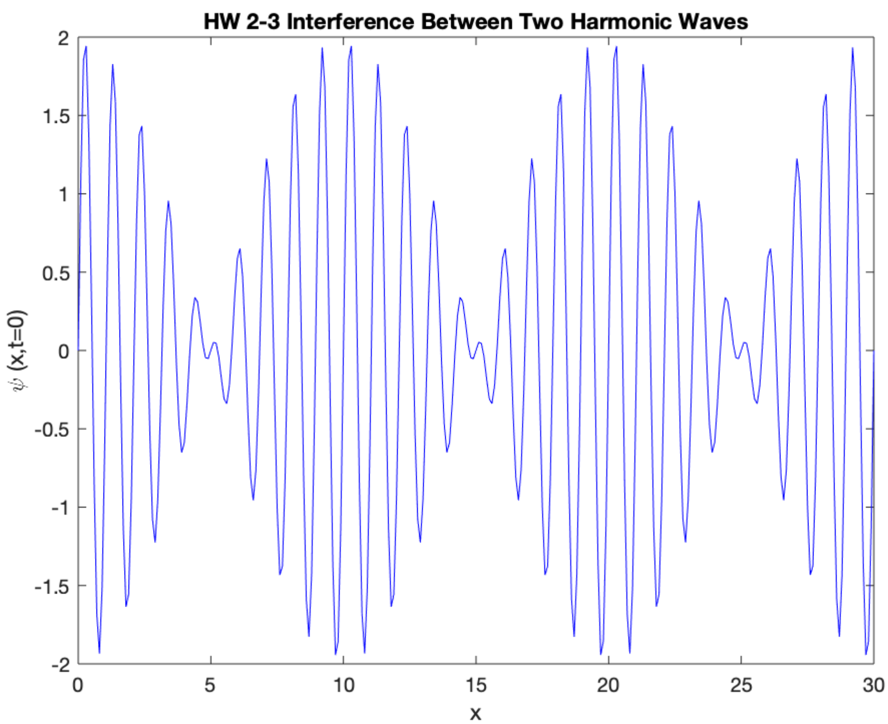
```
clear;clc;

x=-20:0.1:25;
t=[0,15,30];
v=1;

figure(1);
for i=1:length(t)
    y=exp(-(x-15+v*t(i)).^2).*cos(10.*(x+v*t(i)));
    plot(x,y);
    hold on;
end
hold off;

xlabel('x');
ylabel('\psi (x,t)');
legend('t=0','t=15','t=30')
title('HW 2-1 Ultrafast Optical Pulse with Gaussian Envelope');
```



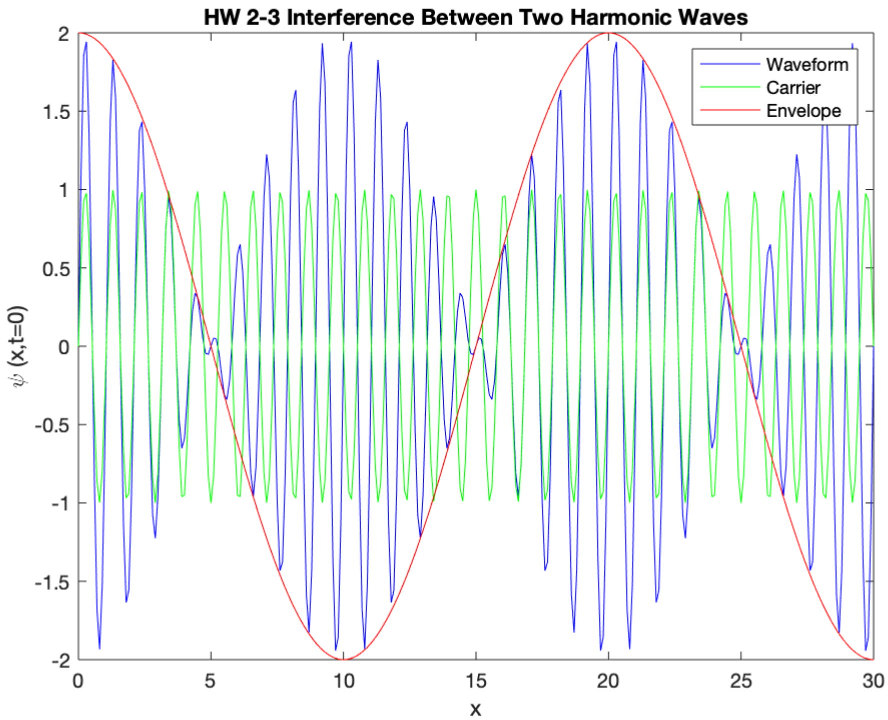

```
%The cosine function is the high frequency carrier, and the exponential is
%the slow varying modulation env
```

HW 2: Problem 3:

```
clearvars;clc;

x=0:0.1:30;
k1=2*pi;
k2=1.8*pi;

y=sin(k1.*x)+sin(k2.*x);
ym=2*cos(0.5*(k1-k2).*x);
yp=sin(0.5*(k1+k2).*x);

figure(2);
plot(x,y,'b');
xlabel('x');
ylabel('\psi (x,t=0)');
title('HW 2-3 Interference Between Two Harmonic Waves');
```




```
figure(3);
plot(x,y,'b',x,yp,'g',x,ym,'r');
xlabel('x');
ylabel('\psi (x,t=0)');
title('HW 2-3 Interference Between Two Harmonic Waves');
legend('Waveform','Carrier','Envelope')
```


HW 2-3 Interference Between Two Harmonic Waves
Waveform
Carrier
Envelope
ψ (x,t=0)
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30